HUNGARIAN MATHEMATICAL OLYMPIAD
2004-2005

Arany Dániel competition (junior olympiad)

grade 9 / final round

1. Prove that 48 is a divisor of $pr(r-p)+rq(q-r)+qp(p-q)$ if p, q, r, are positive integers.

2. Let $ABCD$ be a quadrilateral, $AB=BC=CD=1$, $\angle ABC = 80°$, $\angle BCD = 160°$. Determine the other two angles of $ABCD$.

3. Find the minimal value of the following expression if $x+2y-1=0$:
$$\sqrt{(x-3)^2+(y+1)^2}+\sqrt{(x-7)^2+(y+3)^2}\ .$$

grade 10 / final round

1. The perpendicular bisector of the hypotenouse of a right triangle divides the area in the ratio $1:n$, where n is a positive integer. Find the possible values of the angles of the triangle.

2. Prove that the reciprocal of any positive integer can be written as the sum of the reciprocals of 2005 distinct positive integers.

3. n points are given in the plane ($n>3$). Any three of them can be covered by a triangle of area 1. Prove that we may cover all the points by a triangle of area 4.

grade 10 / final round / specialized mathematical classes

1. ABC is a right triangle with area x. The two different squares inscribed in ABC in such a way that all vertices of both of them lie on the perimeter of ABC have areas y and z respectively. Find the maximal value of $(y+z):x$.

2. There are 1004 boxes on a table. There is either a black or a white ball in each box. We know that the number of white balls is even and there are at least two of them. We may ask the following question: we choose two boxes and we will be told whether a white ball is among the two balls.

   At least how many questions do we need in order to be sure to find two boxes with white balls?

3. On the plane a point P and a line e are given. We have a straight edge and a disk of radius 1 (without its centre). With the disk we may draw a unit circle through any two points A and B if their distance is not greater than 2. How can we construct the mirror image of P with respect to e?

National olympiad, grades 11-12, category II.

Second round

1. Find the real solutions of the following system of equations

$$\sqrt{x+y}+\sqrt{x-y}=10\,; \qquad x^2-y^2-z^2=476\,; \qquad 2^{(\lg|y|-\lg z)}=1.$$

2. ABC is a triangle, the points B_1 and C_1 are on BC, B_2 is on AB, C_2 is on AC. B_1B_2 is parallel to AC, C_1C_2 is parallel to AB. The lines B_1B_2 and C_1C_2 meet at D. Denote the area of BB_1B_2 and CC_1C_2 by b and c respectively.
(a) Prove that if $b=c$, then the centroid of ABC is on the line AD.
(b) Find the value of $b:c$ if D is the incentre of ABC and $AB=4$, $BC=5$, $CA=6$.

3. At each vertex of a pentagon there is a real number. We write on the sides and the diagonals the sum of the numbers which are at the endpoints of them. Out of these 10 numbers we know that 7 are integers. Prove that each of the 10 numbers are integers.

4. The divisors of n are $1 = d_1 < d_2 < ... < d_8 = n$. We know that $20 \le d_6 \le 25$. Find the possible values of n.

Final round

1. The positive integer n is called 'charming' if there are integers $a_1, a_2, \ldots, a_n$ (not necessarily different) such that $a_1 + a_2 + ... + a_n = a_1 \cdot a_2 \cdot .... \cdot a_n = n$. Find the 'charming' integers.

2. The numbers a, b, c are positive and real.
(a) Prove that $\sqrt{\frac{a^2+b^2}{2}} + \frac{2}{\frac{1}{a}+\frac{1}{b}} \ge \frac{a+b}{2} + \sqrt{ab}$.
(b) Is it true always that $\sqrt{\frac{a^2+b^2+c^2}{3}} + \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \ge \frac{a+b+c}{3} + \sqrt[3]{abc}$?

3. ABC is an acute triangle, $\angle BAC = 60°$, $AB=c$, $AC=b$ and $b>c$. The orthocentre and the circumcentre of ABC are M and O respectively. The line OM cuts AB and CA at X and Y respectively.
(a) Prove that the perimeter of AXY is $b+c$. (b) Prove that $OM=b-c$.

First round (specialized mathematical classes), category III.

1. $ABCD$ is a cyclic quadrilateral. Prove that:
$$\frac{AC}{BD} = \frac{DA \cdot AB + BC \cdot CD}{AB \cdot BC + CD \cdot DA}.$$

2. How many real numbers x are there in the interval $0<x<2004$ such that $x + \lfloor x^2 \rfloor = x^2 + \lfloor x \rfloor$. ($\lfloor c \rfloor$ denotes the greatest integer k such that $k \le c$.)

3. Let us denote the sum of the positive divisors of n, excluding n itself by $s(n)$. We call three not necessarily different integers (a, b, c) a friendly triple if $1 < a \le b \le c$ and $s(a)+s(b)=c$, $s(b)+s(c)=a$ and $s(c)+s(a)=b$. Find the friendly triples (a, b, c) where c is even.

4. The set A of different positive integers has k elements. If the positive integers x and y are not contained in A then neither are $2x$, $2y$, and $x+y$. The sum of the elements of A is s. Find the maximal possible value of s.

5. We have a pyramid $ABCDE$, where $ABCD$ is a cyclic quadrilateral. The perpendicular projection of E to the plane $ABCD$ is F. Prove that the perpendicular projections of F to AE, BE, CE, DE are on a circle.

Final round (specialized mathematical classes), category III.

1. *ABCD* is a trapezoid where the sides *AB* and *CD* are parallel. *E* is a point on the side *AB* such that *EC* and *AD* are parallel. The area of the triangle determined by the lines *AC*, *BD*, *DE* is *t*, the area of *ABCD* is *T*. Determine the ratio *AB*:*CD* if *t*:*T* is maximal.

2. Find the greatest integer *k* which satisfies the following property: For every integers *x*, *y* whenever *xy*+1 is divisible by *k*, *x*+*y* is also divisible by *k*.

3. Haydn and Beethoven celebrate the birthday of Mozart with a game. They take numbers alternatly according to the following rule. First Haydn takes the number 2. The next player can take the sum or the product of any two numbers which were taken earlier (it is possible to choose just one number and take it twice or the square of it). The numbers which are taken must be different and smaller than 1757. The winner is that player who can take 1756. Which player has got a winning strategy?

THE PROBLEMS OF THE 2004 KÜRSCHÁK COMPETITION

1. The circle *k* and the circumcircle of the triangle *ABC* are touching externally. The circle *k* is also touching the rays *AB* and *AC* at the points *P* and *Q* respectively. Prove that the midpoint of the segment *PQ* is the centre of the excircle touching the side *BC* of the triangle *ABC*.

2. Find the smallest positive integer *n* different from 2004 with the property that there exists a polynomial $f(x)$ with integer coeficients such that the equation $f(x)=2004$ has at least one integer solution and the equation $f(x)=n$ has at least 2004 distinct integer solutions.

3. Some red and blue points are situated on the circumference of a circle. We can perform the following operations:

(a) we can insert a new red point somewhere and simultaneously change the colours of its two neighbours to the opposite colour.

(b) if there are at least three points, at least one of which is red, then we can delete a red point and simultaneously change the colours of its two neighbours to the opposite colour.

In the beginning there are two points on the circle, both blue. Can we arrive by applying a sequence of the previous two operations, to a position, where we have again two points, both red?